Assignment 1

Posted on 25 Jan 2016 | Clarify doubts by 2 Feb 2016 | Submit by 5 Feb 2016

This is a *Group Assignment* – each group (two students) should submit a single set of solutions.

The solutions may be submitted either as a clearly legible hand-written document, or as a single LaTeX generated PDF document. In case of PDF submission, the filename should be `assign1_groupXX.pdf`, where `XX` is the serial number of the group. Be cogent, but concise.

Attempt all problems. This assignment is worth 150 points in total.

Problem 1 (CS205A 2013, Stanford) [5 + 5 + 20 + 10 = 40]

A. For estimating numerical errors in the process of evaluating $x \times y$ in floating-point arithmetic, which of the following models would you choose to represent the error? Justify your answer.

- Model 1 : We will assume that evaluating $x \times y$ on the computer outputs $(1+\epsilon)(x \times y)$ for some number ϵ satisfying $0 \leq |\epsilon| < \epsilon_{max} \ll 1$, where ϵ may depend upon x, y.
- Model 2 : We will assume that evaluating $x \times y$ on the computer outputs $(x \times y) + \epsilon$ for some number ϵ satisfying $0 \leq |\epsilon| < \epsilon_{max} \ll 1$, where ϵ may depend upon x, y.

B. Suppose that $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ satisfy $0 \leq |\epsilon_i| < \epsilon_{max} \ll 1$ for all $i = 1, 2, \ldots, k$. Prove that there exists some ϵ satisfying $0 \leq |\epsilon| < \epsilon_{max} \ll 1$ such that $(1+\epsilon_1)(1+\epsilon_2)\cdots(1+\epsilon_k) = (1+\epsilon)^k$.

C. Suppose we want to compute $x \times y$. Assume that the process introduces a numerical error ϵ satisfying $0 \leq |\epsilon| < \epsilon_{max} \ll 1$, as per the model you chose earlier. Moreover, in reality, we do not know the inputs x, y accurately – we just know them relative to the numerical precision. Thus, let us denote the inputs by $(1+\epsilon_x)\ x$ and $(1+\epsilon_y)\ y$, where $0 \leq |\epsilon_x|, |\epsilon_y| < \epsilon_{max} \ll 1$. Compute the bounds for error while evaluating $x \times y$, in terms of ϵ_{max}.

D. In a similar fashion, compute the bounds for error while evaluating $(x - y)$, in terms of ϵ_{max}.

Problem 2 (CS205A 2013, Stanford) [20 + 20 = 40]

A. Suppose we want to evaluate nx (where $n \ll 1/\epsilon_{max}$) using the recurrence

$$S_1 = x$$
$$S_n = S_{n-1} + x$$

Compute a bound for the relative error in this computation, in terms of n and ϵ_{max}.

B. Is there a way to evaluate nx with the bound on the relative error *not* linearly dependent on n? Describe such a method, if it exists, and compute the bound for the relative error.

Problem 3 $[(4 \times 5) + 20 = 40]$

A. Solve the following system of linear equations for x_1, x_2, x_3, x_4, using each of the methods prescribed below. In each case, count (precisely) the number of individual operations (additions, subtractions, multiplications and divisions) required to solve the system.

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ 4x_1 + 3x_2 + 3x_3 + x_4 &= -3 \\ 8x_1 + 7x_2 + 9x_3 + 5x_4 &= 3 \\ 6x_1 + 7x_2 + 9x_3 + 8x_4 &= 2 \end{aligned}$$

(i) Using simple Gaussian elimination with *nonzero* pivoting.

(ii) Using simple Gaussian elimination with *partial* pivoting.

(iii) Using PLU factorization technique, with *nonzero* pivoting.

(iv) Using PLU factorization technique, with *partial* pivoting.

B. Estimate the number of operations required to solve a system of n linear equations with n unknowns, for each of the above methods. The estimates should be functions of n.

Problem 4 $[10 + 10 + 10 = 30]$

A. Solve the following system of linear equations for x_1, x_2, x_3, x_4, using any suitable computational method of your choice. You may use any result you have derived thus far.

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 + i \\ 8x_1 + 7x_2 + 9x_3 + 5x_4 &= 3 + 2i \\ 8x_1 + 6x_2 + 6x_3 + 2x_4 &= -6 \\ 6x_1 + 7x_2 + 9x_3 + 8x_4 &= 2 - 3i \end{aligned}$$

B. Propose an algorithm to solve a system of complex linear equations $\mathbf{A}(\vec{x} + i\vec{y}) = (\vec{u} + i\vec{v})$? Estimate the number of operations as a function of n, where $\mathbf{A}$ is $n \times n$.

C. Propose an algorithm to solve a system of complex linear equations $(\mathbf{A} + i\mathbf{B})(\vec{x} + i\vec{y}) = (\vec{u} + i\vec{v})$? Estimate the number of operations as a function of n, where $\mathbf{A}, \mathbf{B}$ are $n \times n$.